

Impedance Transmission Conditions for the Electric Potential across a Highly Conductive Casing

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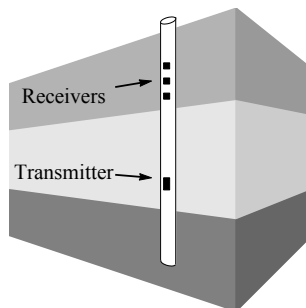
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MOTIVATION

- **Main goal:** To obtain a better characterization of the Earth's subsurface
- **How:** Recording borehole resistivity measurements
- **Procedure:**
 - Well
 - Transmitters
 - Receivers



MOTIVATION

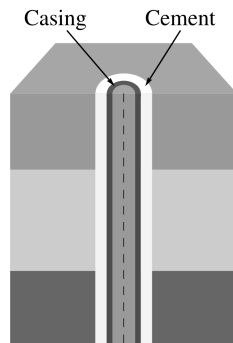
- **Practical Difficulties:**

- It is not easy to drill a borehole
- It may collapse

- **Practical Solutions:**

- Use a metallic casing
- Surround with a cement layer

- **Problem solved, but...**



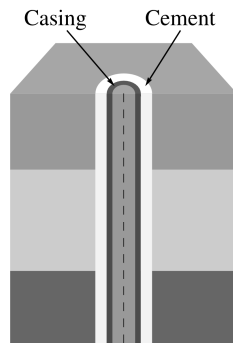
MOTIVATION

- **Practical Difficulties:**

- It is not easy to drill a borehole
- It may collapse

- **Practical Solutions:**

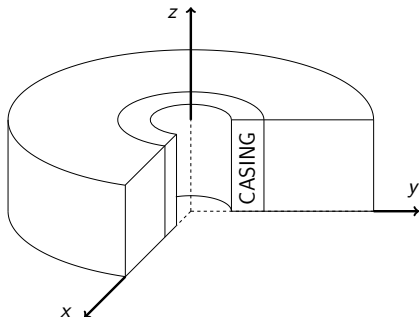
- Use a metallic casing
- Surround with a cement layer



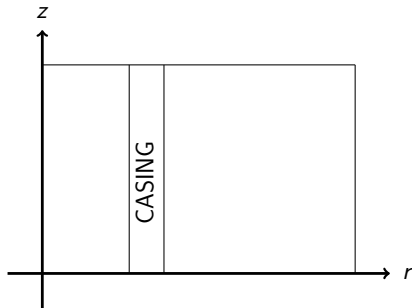
- **Problem solved, but... Numerical problems due to the high conductivity and thinness of the casing**

CONFIGURATION OF INTEREST

SECTIONED 3D DOMAIN



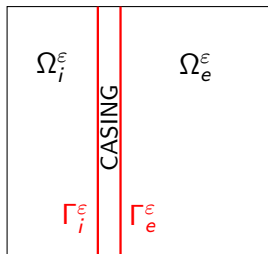
MERIDIAN DOMAIN



We reduce the 3D problem to a 2D problem due to the **axysimmetric** configuration.

MAIN IDEA

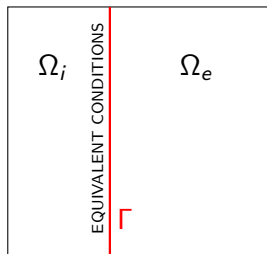
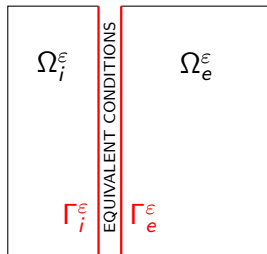
REFERENCE MODEL



1st Approach

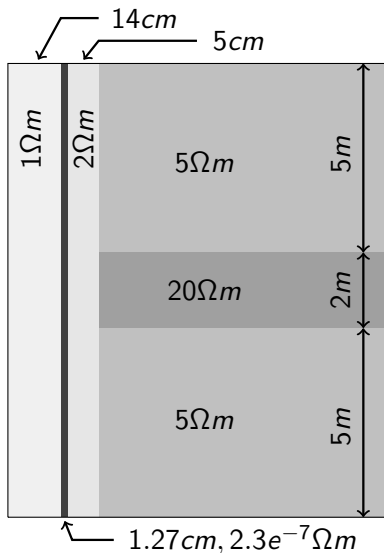
2nd Approach

ASYMPTOTIC MODELS



Idea: Replace the casing by
equivalent conditions.

REALISTIC SCENARIO



- Conductivity and casing width:**

$$\begin{cases} \varepsilon &= 1.27e - 2m \\ \sigma_c &= 4.34e6 \Omega^{-1}m^{-1} \end{cases}$$

$$\Rightarrow \sigma_c \approx \varepsilon^{-3}$$

- Classical approach:**

$$\sigma_c = \alpha \quad \alpha \in \mathbb{R}$$

- High conductive case:**

$$\sigma_c = \alpha \varepsilon^{-3} \quad \alpha \in \mathbb{R}$$

REFERENCES

- [1] A.A. Kaufman. The electrical field in a borehole with a casing. *Geophysics*, Vol.55, Issue 1, pp. 29-38, 1990.
- [2] D.Pardo, C.Torres-Verdín and Z.Zhang. Sensitivity study of borehole-to-surface and crosswell electromagnetic measurements acquired with energized steel casing to water displacement in hydrocarbon-bearing layers. *Geophysics*, 73 No.6, F261-F268, 2008.
- [3] M. Duruflé, V. Péron and C. Poignard. Thin Layer Models for Electromagnetism. *Communications in Computational Physics* 16(1):213-238, 2014.
- [4] K. Schmidt, A. Chernov, Robust transmission conditions of high order for thin conducting sheets in two dimensions, *IEEE Trans. Magn.*, 50 (2014), pp. 41–44.

OUTLINE

- 1 Equivalent Conditions
- 2 Numerical Results
- 3 Second Approach
- 4 Application
- 5 Perspectives

OUTLINE

- 1 **Equivalent Conditions**
- 2 Numerical Results
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MODEL PROBLEM - 3D

STATIC ELECTRIC POTENTIAL

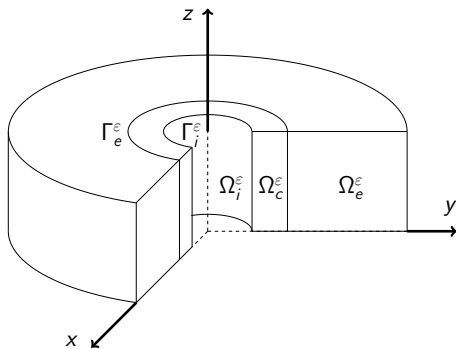
$$\operatorname{div}((\sigma - i\epsilon\omega) \nabla u) = f \text{ in } \Omega$$

Where f is a known data, σ is piecewise constant

$$\sigma = \begin{cases} \sigma_i & \text{in } \Omega_i^\epsilon \\ \sigma_c = \alpha\epsilon^{-3} & \text{in } \Omega_c^\epsilon \\ \sigma_e & \text{in } \Omega_e^\epsilon \end{cases}$$

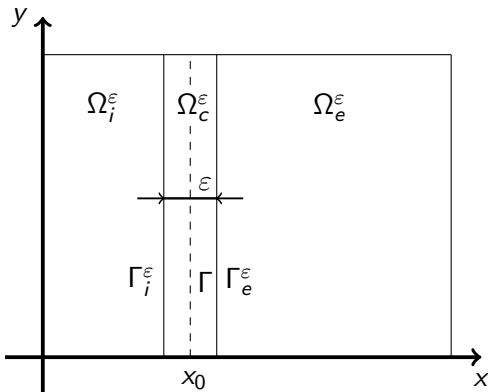
and the solution is expressed as

$$u = \begin{cases} u_i & \text{in } \Omega_i^\epsilon \\ u_c & \text{in } \Omega_c^\epsilon \\ u_e & \text{in } \Omega_e^\epsilon \end{cases}$$



MODEL PROBLEM - 2D

Reference problem (P) set in the domain $\Omega = \Omega_i^\varepsilon \cup \Omega_c^\varepsilon \cup \Omega_e^\varepsilon \cup \Gamma_i^\varepsilon \cup \Gamma_e^\varepsilon$.



$$\left\{ \begin{array}{lll} \sigma_i \Delta u_i & = f_i & \text{in } \Omega_i^\varepsilon \\ \sigma_c \Delta u_c & = 0 & \text{in } \Omega_c^\varepsilon \\ \sigma_e \Delta u_e & = f_e & \text{in } \Omega_e^\varepsilon \\ u_i & = u_c & \text{on } \Gamma_i^\varepsilon \\ u_c & = u_e & \text{on } \Gamma_e^\varepsilon \\ \sigma_i \partial_n u_i & = \sigma_c \partial_n u_c & \text{on } \Gamma_i^\varepsilon \\ \sigma_c \partial_n u_c & = \sigma_e \partial_n u_e & \text{on } \Gamma_e^\varepsilon \\ u & = 0 & \text{on } \partial\Omega \end{array} \right.$$

EXISTENCE AND UNIQUENESS

Theorem 1

There exists $\varepsilon_0 > 0$ s.t. for all $\varepsilon \in (0, \varepsilon_0)$, if $f_i \in L^2(\Omega_i^\varepsilon)$ and $f_e \in L^2(\Omega_e^\varepsilon)$, then $\exists! u \in H_0^1(\Omega)$ solution of (P) and

$$\|u\|_{1,\Omega} \leq C \left(\|f_i\|_{0,\Omega_i^\varepsilon} + \|f_e\|_{0,\Omega_e^\varepsilon} \right).$$

Proof:

- Derive **variational formulation**, for all $v \in H_0^1(\Omega)$, $a(u, v) = b(v)$.
- Prove that a is **coercive** in $H_0^1(\Omega)$.
- Apply the **Lax-Milgram** Lemma.

DEFINITIONS

Definition 1

We define the **jump** and **mean value** of the solution u across the casing as

$$[u] = u|_{\Gamma_e^\varepsilon} - u|_{\Gamma_i^\varepsilon}$$

$$\{u\} = \frac{1}{2} \left(u|_{\Gamma_e^\varepsilon} + u|_{\Gamma_i^\varepsilon} \right)$$

Definition 2

Let u be the reference solution. We say an asymptotic model is of **Order $k+1$** , if its solution $u^{[k]}$ satisfies

$$\|u - u^{[k]}\|_{H^1} \leq C\varepsilon^{k+1} \quad \text{when} \quad \varepsilon \rightarrow 0$$

METHODOLOGY

- **Step1:** Derive an **Asymptotic Expansion** for u when $\varepsilon \rightarrow 0$

- In the casing:
$$u_c(x, y) = \sum_{n \in \mathbb{N}} \varepsilon^n U^n \left(\frac{x - x_0}{\varepsilon}, y \right)$$

- Outside the casing:
$$u(x, y) = \sum_{n \in \mathbb{N}} \varepsilon^n u^n(x, y)$$

- **Step2:** Obtain **Equivalent Conditions** of order $k + 1$ by identifying a simpler problem satisfied by the truncated expansion and neglecting the higher order terms in ε

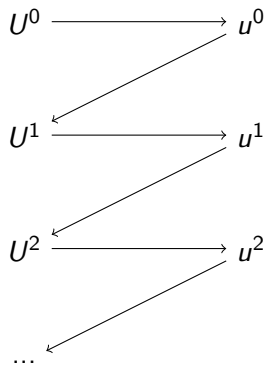
- $$u_{k, \varepsilon} := u^0 + \varepsilon u^1 + \varepsilon^2 u^2 + \dots + \varepsilon^k u^k$$

- **Step3:** **Prove Convergence** rates for asymptotic models

MULTISCALE EXPANSION

$$\left\{ \begin{array}{ll} \partial_X^2 U^k + \partial_y^2 U^{k-2} = 0 & \text{in } \Omega_c^\varepsilon \\ \sigma_i \Delta u_i^k = f_i & \text{in } \Omega_i^\varepsilon \\ \sigma_e \Delta u_e^k = f_e & \text{in } \Omega_e^\varepsilon \\ u_i^k = U^k & \text{on } \Gamma_i^\varepsilon \\ U^k = u_e^k & \text{on } \Gamma_e^\varepsilon \\ \sigma_i \partial_X u_i^{k-4} = \alpha \partial_X U^k & \text{on } \Gamma_i^\varepsilon \\ \alpha \partial_X U^k = \sigma_e \partial_X u_e^{k-4} & \text{on } \Gamma_e^\varepsilon \\ U^k = 0 & \text{on } \partial\Omega \end{array} \right.$$

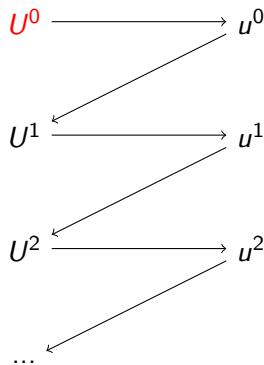
PROCEDURE



MULTISCALE EXPANSION

$$\left\{ \begin{array}{ll} \partial_X^2 U^k + \partial_y^2 U^{k-2} = 0 & \text{in } \Omega_c^\varepsilon \\ \sigma_i \Delta u_i^k = f_i & \text{in } \Omega_i^\varepsilon \\ \sigma_e \Delta u_e^k = f_e & \text{in } \Omega_e^\varepsilon \\ u_i^k = U^k & \text{on } \Gamma_i^\varepsilon \\ U^k = u_e^k & \text{on } \Gamma_e^\varepsilon \\ \sigma_i \partial_X u_i^{k-4} = \alpha \partial_X U^k & \text{on } \Gamma_i^\varepsilon \\ \alpha \partial_X U^k = \sigma_e \partial_X u_e^{k-4} & \text{on } \Gamma_e^\varepsilon \\ U^k = 0 & \text{on } \partial\Omega \end{array} \right.$$

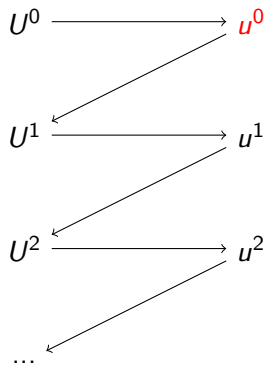
PROCEDURE



MULTISCALE EXPANSION

$$\left\{ \begin{array}{ll} \partial_X^2 U^k + \partial_y^2 U^{k-2} = 0 & \text{in } \Omega_c^\varepsilon \\ \sigma_i \Delta \mathbf{u}_i^k = f_i & \text{in } \Omega_i^\varepsilon \\ \sigma_e \Delta \mathbf{u}_e^k = f_e & \text{in } \Omega_e^\varepsilon \\ \mathbf{u}_i^k = U^k & \text{on } \Gamma_i^\varepsilon \\ U^k = \mathbf{u}_e^k & \text{on } \Gamma_e^\varepsilon \\ \sigma_i \partial_X u_i^{k-4} = \alpha \partial_X U^k & \text{on } \Gamma_i^\varepsilon \\ \alpha \partial_X U^k = \sigma_e \partial_X u_e^{k-4} & \text{on } \Gamma_e^\varepsilon \\ U^k = 0 & \text{on } \partial\Omega \end{array} \right.$$

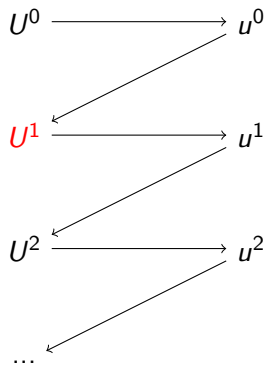
PROCEDURE



MULTISCALE EXPANSION

$$\left\{ \begin{array}{ll} \partial_X^2 \mathbf{U}^k + \partial_y^2 U^{k-2} = 0 & \text{in } \Omega_c^\varepsilon \\ \sigma_i \Delta u_i^k = f_i & \text{in } \Omega_i^\varepsilon \\ \sigma_e \Delta u_e^k = f_e & \text{in } \Omega_e^\varepsilon \\ u_i^k = \mathbf{U}^k & \text{on } \Gamma_i^\varepsilon \\ \mathbf{U}^k = u_e^k & \text{on } \Gamma_e^\varepsilon \\ \sigma_i \partial_x u_i^{k-4} = \alpha \partial_X \mathbf{U}^k & \text{on } \Gamma_i^\varepsilon \\ \alpha \partial_X \mathbf{U}^k = \sigma_e \partial_x u_e^{k-4} & \text{on } \Gamma_e^\varepsilon \\ \mathbf{U}^k = 0 & \text{on } \partial\Omega \end{array} \right.$$

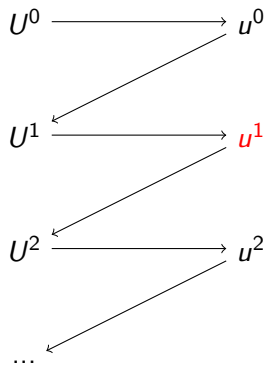
PROCEDURE



MULTISCALE EXPANSION

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PROCEDURE



EQUIVALENT MODELS

FIRST APPROACH

(Transmission conditions across the casing)

• **Order 2:**

$$\begin{cases} \sigma_i \Delta u_i = f_i & \text{in } \Omega_i^\varepsilon \\ u_i = 0 & \text{on } \partial\Omega_i^\varepsilon \end{cases} \quad (P_2)$$

$$\begin{cases} \sigma_e \Delta u_e = f_e & \text{in } \Omega_e^\varepsilon \\ u_e = 0 & \text{on } \partial\Omega_e^\varepsilon \end{cases}$$

• **Order 4:**

$$\begin{cases} \sigma_i \Delta u_i = f_i & \text{in } \Omega_i^\varepsilon \\ \sigma_e \Delta u_e = f_e & \text{in } \Omega_e^\varepsilon \\ [u] = 0 \\ [\sigma \partial_n u] = -\frac{\alpha}{\varepsilon^2} \Delta_{\Gamma^\varepsilon} \{u\} \\ u = 0 & \text{on } \partial\Omega \cup \partial(\Omega_e^\varepsilon \cup \Omega_i^\varepsilon) \end{cases} \quad (P_4)$$

CONVERGENCE

Theorem 2

There exists $\varepsilon_0 > 0$ s.t. for all $\varepsilon \in (0, \varepsilon_0)$, if $f_i \in L^2(\Omega_i^\varepsilon)$ and $f_e \in L^2(\Omega_e^\varepsilon)$, then $\exists! u^{[k]} \in V_{k+1}$ solution of (P_{k+1}) , $k = 1, 3$, and

$$\|u^{[k]}\|_{1, \Omega_i^\varepsilon \cup \Omega_e^\varepsilon} \leq C \left(\|f_i\|_{0, \Omega_i^\varepsilon} + \|f_e\|_{0, \Omega_e^\varepsilon} \right)$$

$$\|u - u^{[k]}\|_{1, \Omega_i^\varepsilon \cup \Omega_e^\varepsilon} \leq C \varepsilon^{k+1}$$

Where

$$V_2 = \{v : v_i \in H_0^1(\Omega_i^\varepsilon), v_e \in H_0^1(\Omega_e^\varepsilon)\}$$

$$V_4 = \{v : v_i \in H^1(\Omega_i^\varepsilon), v_e \in H^1(\Omega_e^\varepsilon), \nabla_{\Gamma^\varepsilon} \{v\} \in L^2(\Gamma^\varepsilon), v|_{\Gamma_i^\varepsilon} = v|_{\Gamma_e^\varepsilon}, v|_{\Gamma_0} = 0\}$$

OUTLINE

1 Equivalent Conditions

2 Numerical Results

3 Second Approach

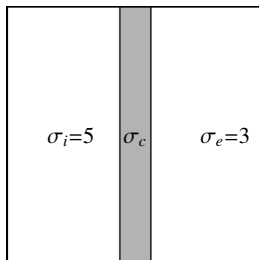
4 Application

5 Perspectives

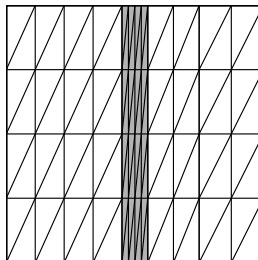
FEM CODE

- Classic Finite Element Method code
- Straight triangular elements (h refinement)
- Lagrange shape functions of any degree (p refinement)

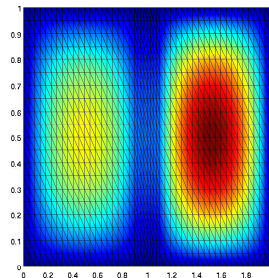
Domain



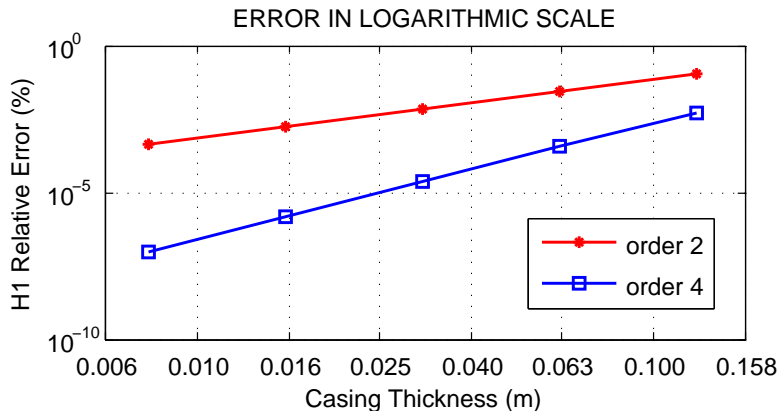
Mesh



Solution



CONVERGENCE RATES



Casing Thickness (ε)	0.0117	0.0234	0.0469	0.0938	Expected ($\varepsilon \rightarrow 0$)
Order 2 Slopes	1.9969	1.9899	1.9638	1.8624	2
Order 4 Slopes	4.0046	4.0039	3.9879	3.8992	4

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EQUIVALENT MODELS

SECOND APPROACH

(Transmission conditions across the interface Γ)

• **Order 1:**

$$\begin{cases} \sigma_i \Delta u_i = f_i & \text{in } \Omega_i \\ u_i = 0 & \text{on } \partial\Omega_i \\ \sigma_e \Delta u_e = f_e & \text{in } \Omega_e \\ u_e = 0 & \text{on } \partial\Omega_e \end{cases} \quad (P_1)$$

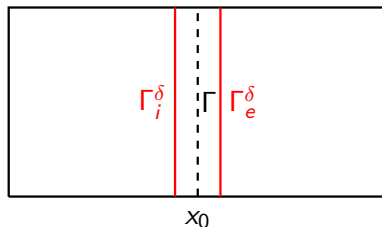
• **Order 2:**

$$\begin{cases} \sigma_i \Delta u_i = f_i & \text{in } \Omega_i \\ \sigma_e \Delta u_e = f_e & \text{in } \Omega_e \\ [u] = -\varepsilon \{\partial_n u\} & \text{on } \Gamma \\ [\partial_n u] = -\frac{4}{\varepsilon} \{u\} & \text{on } \Gamma \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

STABILITY

Problem

Stability problems for the order 2 model of the second approach due to a non coercive term.



Solution

Use of Artificial Boundaries to move the transmission conditions and recover stability. For $\delta > 0.5$

$$\Gamma_i^\delta = \{(x, y) : x = x_0 - \delta\varepsilon, y \in (0, y_0)\}$$

$$\Gamma_e^\delta = \{(x, y) : x = x_0 + \delta\varepsilon, y \in (0, y_0)\}$$

EQUIVALENT MODELS

SECOND APPROACH

(Transmission conditions across the artificial interfaces)

$$\bullet \text{ Order 2: } \left\{ \begin{array}{ll} \sigma_i \Delta u_i = f_i & \text{in } \Omega_i \\ \sigma_e \Delta u_e = f_e & \text{in } \Omega_e \\ [u] = -\varepsilon (1 - 2\delta) \{ \partial_n u \} & \\ [\partial_n u] = -\frac{4(1 - 2\delta)}{\varepsilon} \{ u \} & \\ u = 0 & \text{on } \partial\Omega \end{array} \right. \quad (P_2)$$

Stable if $\delta > 0.5$

CONVERGENCE

Theorem 3

There exists $\varepsilon_0 > 0$ s.t. for all $\varepsilon \in (0, \varepsilon_0)$, if $f_i \in L^2(\Omega_i^\varepsilon)$ and $f_e \in L^2(\Omega_e^\varepsilon)$, then $\exists! u^{[k]} \in V_{k+1}$ solution of (P_{k+1}) , $k = 1, 2$, and

$$\|u^{[k]}\|_{1, \Omega_i^\varepsilon \cup \Omega_e^\varepsilon} \leq C \left(\|f_i\|_{0, \Omega_i^\varepsilon} + \|f_e\|_{0, \Omega_e^\varepsilon} \right)$$

$$\|u - u^{[k]}\|_{1, \Omega_i^\varepsilon \cup \Omega_e^\varepsilon} \leq C \varepsilon^{k+1}$$

Where

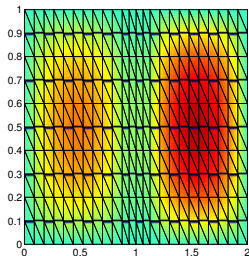
$$V_1 = \{v : v_i \in H_0^1(\Omega_i), v_e \in H_0^1(\Omega_e)\}$$

$$V_2 = \{v : v_i \in H^1(\Omega_i), v_e \in H^1(\Omega_e), v_i|_\Gamma = v_e|_\Gamma, v|_{\Gamma_0} = 0\}$$

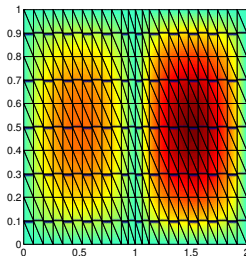
QUALITATIVE COMPARISON

X-Y VIEW

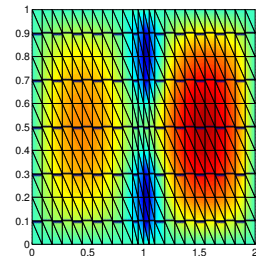
Reference Model



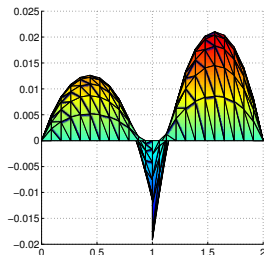
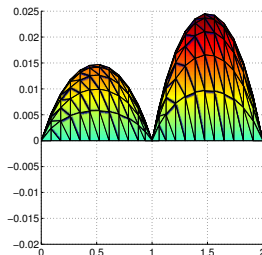
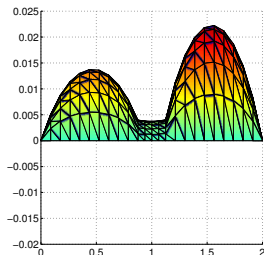
Order 1 Model



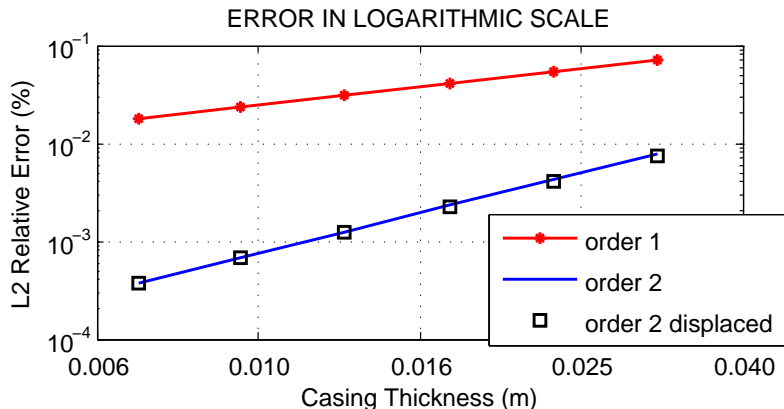
Order 2 Model



X-Z VIEW

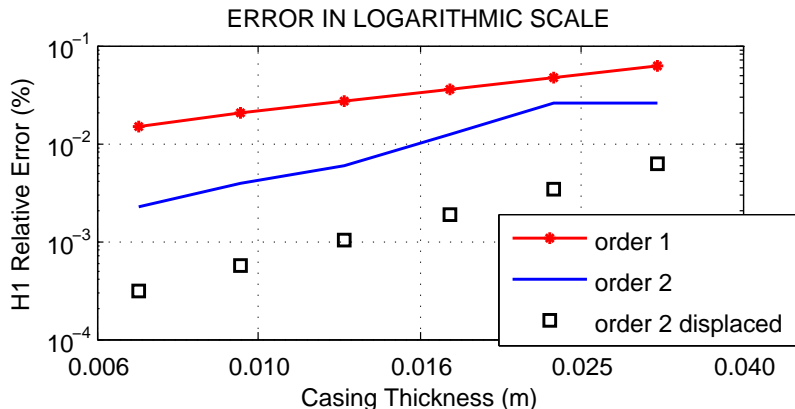


CONVERGENCE RATES L^2



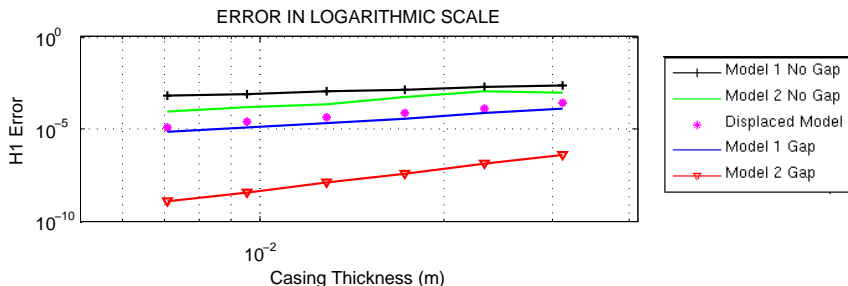
Casing Thickness (ε)	0.008	0.011	0.015	0.020	0.027	Expected ($\varepsilon \rightarrow 0$)
Order 1 Slopes	0.972	0.962	0.948	0.928	0.899	1
Order 2 Slopes	2.032	2.011	2.062	2.090	1.9636	2
Order 2 displaced	2.005	2.006	2.007	2.008	2.007	2

CONVERGENCE RATES H^1



Casing Thickness (ε)	0.008	0.011	0.015	0.020	0.027	Expected ($\varepsilon \rightarrow 0$)
Order 1 Slopes	0.982	0.975	0.966	0.953	0.935	1
Order 2 Slopes	1.915	1.302	2.463	2.569	-0.125	2
Order 2 displaced	2.004	2.005	2.006	2.008	2.007	2

COMPARISON

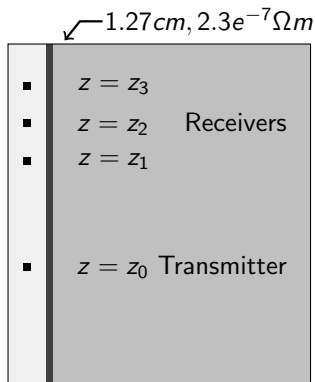


	Order	Stability	ε -independent Domain
Model 1 No Gap	1	✓	✓
Model 2 No Gap	2	✗	✓
Displaced Model	2	✓	✗
Model 1 Gap	2	✓	✗
Model 2 Gap	4	✓	✗

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APPLICATION



• **Equation:** $\text{div}(\sigma \nabla u) = f$

• **Right hand side:**

$$f = \begin{cases} 1 & \text{In the transmitter} \\ 0 & \text{Outside the transmitter} \end{cases}$$

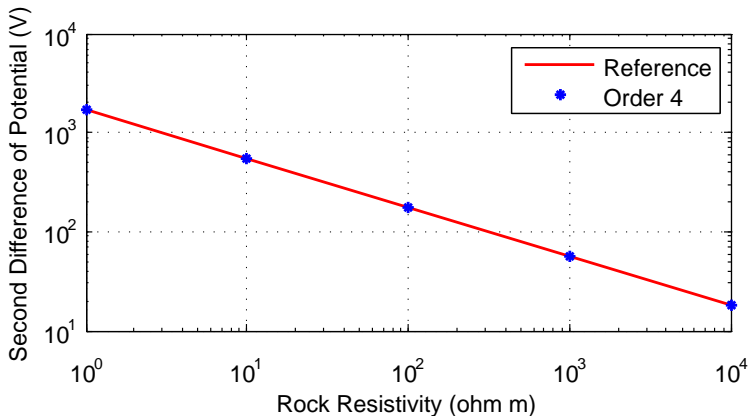
• **Objective:** Measure the second difference of potential on the Receivers

$$U_2 = u(z_1) - 2u(z_2) + u(z_3)$$

• **Expected Result:** (Model of Kaufman) Relation between second difference of potential and rock resistivity of the form

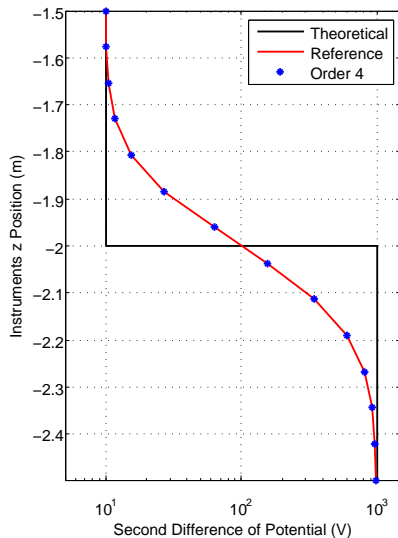
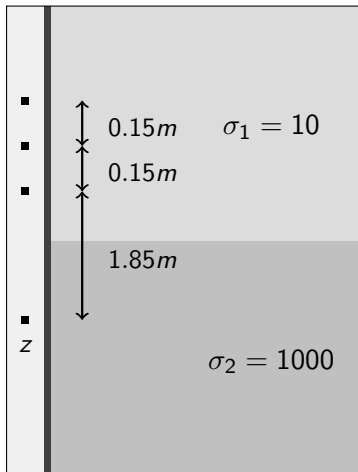
$$U_2 = k \cdot \rho_{\text{rock}}^{-\frac{1}{2}} \quad k \in \mathbb{R}$$

VARYING ROCK CONDUCTIVITY



Resistivity	[1,10]	[10,10 ²]	[10 ² ,10 ³]	[10 ³ ,10 ⁴]	Expected
Reference Model Slopes	-0.4914	-0.4924	-0.4978	-0.4991	-0.5
Order 4 Slopes	-0.4914	-0.4924	-0.4977	-0.4993	-0.5

TWO ROCK LAYERS



OUTLINE

- 1 Equivalent Conditions
- 2 Numerical Results
- 3 Second Approach
- 4 Application
- 5 Perspectives**

Perspectives

- Obtain semianalytical solutions to reduce the computational cost.
- Consider physically more realistic scenarios.
- Develop 3D electromagnetic models.
- Study highly deviated boreholes.

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**THANK YOU FOR YOUR
ATTENTION**